

# TWO-WEIGHT INEQUALITY FOR HOMOGENEOUS SINGULAR INTEGRAL OPERATORS

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(Received March 27, 2000)

**Abstract.** Sufficient conditions on the weights  $u(\cdot)$  and  $v(\cdot)$  are given in order that a homogeneous singular integral operator is bounded from the weighted Lebesgue space  $L^p(v(x)dx)$  to  $L^p(u(x)dx)$  for  $1 < p < \infty$ .

*AMS 1991 Mathematics Subject Classification.* 42B20

*Key words and phrases.* Weighted inequalities, Homogeneous singular integrals

## §1 Introduction

Throughout this paper  $T = T_\Omega$  denotes the *homogeneous singular integral operator* defined on  $\mathbb{R}^n$ ,  $n \geq 1$ , by

$$(Tf)(x) = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < |x-y|} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy.$$

Here  $\Omega(\cdot)$  is a positively homogeneous function of degree zero (that is  $\Omega(\lambda x) = \Omega(x)$  for  $\lambda > 0$ ) with integral value zero on the unit sphere  $S^{n-1}$ .

Our purpose in the present work is to *derive sufficient conditions* on weights  $u(\cdot)$  and  $v(\cdot)$  (i.e. nonnegative and locally integrable functions) *for which*  $T$  *is bounded* from the weighted Lebesgue space  $L^p(v(x)dx)$  to  $L^p(u(x)dx)$ ,  $1 < p < \infty$ . This means that for some constant  $C > 0$

$$(1.1) \quad \int_{\mathbb{R}^n} |(Tf)(x)|^p u(x) dx \leq C^p \int_{\mathbb{R}^n} |f(x)|^p v(x) dx$$

for all smooth functions  $f(\cdot)$ , say  $f(\cdot) \in C_0^\infty(\mathbb{R}^n)$ . For convenience, inequality (1.1) will be also denoted by  $T : L^p(v(x)dx) \rightarrow L^p(u(x)dx)$ .

The unweighted boundedness

$$(1.2) \quad T : L^p(dx) \rightarrow L^p(dx)$$

is known to be true [Ca-Zy] whenever  $\Omega(\cdot)$  is an odd function and integrable on  $S^{n-1}$ . For general  $\Omega(\cdot)$  the situation is more involved. For example from

[Ca-Zy], it is known that (1.2) holds if  $\Omega(\cdot)$  is an even function and  $|\Omega(\cdot)|(1 + \ln^+ |\Omega(\cdot)|)$  integrable on  $S^{n-1}$ . Alternative conditions implying (1.2) can be found in [Gr-St].

The boundedness with one (power) weight

$$(1.3) \quad T : L^p(|x|^\beta dx) \rightarrow L^p(|x|^\beta dx)$$

has been investigated by many authors [Wl], [Du], [Ho1], [Mu-Wh], [Wa], [Wa-Wh]. For  $|\Omega(\cdot)|(1 + \ln^+ |\Omega(\cdot)|) \in L^1(S^{n-1})$  and  $\Omega(\cdot) \in L^r(S^{n-1})$ ,  $r > 1$ , Walsh [Wl] proved that (1.3) holds whenever

$$(1.4) \quad \max\left\{-n, -\frac{np}{r'}\right\} < \beta < \min\left\{n(p-1), \frac{np}{r'}\right\}$$

where  $r' = \frac{r}{r-1}$ . A different approach of this result has been obtained later by D. Watson [Wa]. B. Muckenhoupt and R. Wheeden [Mu-Wh] proved that (1.3) holds under

$$(1.5) \quad \max\left\{-n, -1 - \frac{(n-1)p}{r'}\right\} < \beta < \min\left\{n(p-1), (p-1) + \frac{(n-1)p}{r'}\right\}.$$

Another proof of (1.3), using assumption (1.5), was also found by J. Duoandikoetxea [Du].

Since

$$\frac{np}{r'} \leq (p-1) + \frac{(n-1)p}{r'} \leq n(p-1) \quad \text{for} \quad r \leq p$$

and

$$-n \leq -1 - \frac{(n-1)p}{r'} \leq -\frac{np}{r'} \quad \text{for} \quad p \leq r',$$

the result with assumption (1.5) appears to be better than with (1.4). Indeed if  $1 < r < 2$  and  $r < p < r'$  then  $-n < -1 - \frac{(n-1)p}{r'} < -\frac{np}{r'} < \frac{np}{r'} < (p-1) + \frac{(n-1)p}{r'} < n(p-1)$ . Except for  $2 \leq r < \infty$  and  $r' \leq p \leq r$  for which both (1.5) and (1.4) mean  $-n < \beta < n(p-1)$ , there are more  $\beta$ 's which are defined by (1.5) than by (1.4).

Note that (1.5) is a condensed formula which, after using the above observation, must be understood as grouping six possibilities. Precisely if  $1 \leq r \leq 2$  then (1.5) can be read as

$$(1.6) \quad -1 - \frac{(n-1)p}{r'} < \beta < n(p-1) \quad \text{for} \quad p \leq r \leq r'$$

$$(1.7) \quad -1 - \frac{(n-1)p}{r'} < \beta < (p-1) + \frac{(n-1)p}{r'} \quad \text{for} \quad r < p \leq r'$$

$$(1.8) \quad -n < \beta < (p-1) + \frac{(n-1)p}{r'} \quad \text{for } r \leq r' < p.$$

And if  $2 < r \leq \infty$  then formula (1.5) means that

$$(1.9) \quad -1 - \frac{(n-1)p}{r'} < \beta < n(p-1) \quad \text{for } p \leq r' < r$$

$$(1.10) \quad -n < \beta < n(p-1) \quad \text{for } r' < p \leq r$$

and

$$(1.11) \quad -n < \beta < (p-1) + \frac{(n-1)p}{r'} \quad \text{for } r' < r < p.$$

It follows from these details that for  $1 \leq r < \infty$  the range of power  $\beta$ 's highly depends on the index  $r$ . This fact is already a clue on the difficulty in obtaining inequality (1.1) for weights other than power ones.

The one-weight boundedness

$$(1.12) \quad T : L^p(v(x)dx) \rightarrow L^p(v(x)dx)$$

was first investigated by D. Kurtz and R. Wheeden [Ku-Wh] and the result was improved by J. Duoandikoetxea [Du] and also independently by D. Watson [Wa]. From the papers by these last two authors, it is known that (1.12) holds whenever one of the following situations arises:

$$(1.13) \quad v(\cdot) \in A_{\frac{p}{r'}} \quad \text{for } r' \leq p$$

$$(1.14) \quad v^{1-p'}(\cdot) \in A_{\frac{p'}{r'}} \quad \text{for } p \leq r$$

$$(1.15) \quad v^{r'}(\cdot) \in A_p.$$

Recall that  $w(\cdot) \in A_t$ ,  $1 < t < \infty$ , if for some constant  $A > 0$

$$\left( |Q|^{-1} \int_Q w(y) dy \right)^{\frac{1}{t}} \left( |Q|^{-1} \int_Q w^{1-t'}(y) dy \right)^{\frac{1}{t'}} \leq A \quad \text{for all cubes } Q$$

with sides parallel to the coordinates axes. Here  $t' = \frac{t}{t-1}$ . And  $w(\cdot) \in A_1$  means that  $|Q|^{-1} \int_Q w(y) dy \leq A \times \text{essinf}_Q w(z)$  for all cubes  $Q$ .

The boundedness (1.12) cannot be decided from assumptions (1.13) and (1.14) whenever  $r < p < r'$  and  $1 < r < 2$ . For this situation, the last

alternative issue is the use of (1.15). Unfortunately this criterion is not always applicable. Indeed for  $v(x) = |x|^\beta$ , then (1.15) is that  $-\frac{n}{r'} < \beta < \frac{n(p-1)}{r'}$ . But, in view of (1.7), the boundedness (1.12) holds whenever  $-\frac{n}{r'} - \left[ \left(1 - \frac{p}{r'}\right) + \frac{n(p-1)}{r'} \right] < \beta < \frac{n(p-1)}{r'} + \left[ \left(\frac{p}{r} - 1\right) + \frac{n}{r'} \right]$ . It means that none of criteria (1.13), (1.14) and (1.15) can be used to derive  $T : L^p(|x|^\beta dx) \rightarrow L^p(|x|^\beta dx)$  for all  $\beta$ 's such that  $\frac{n(p-1)}{r'} \leq \beta < \frac{n(p-1)}{r'} + \left[ \left(\frac{p}{r} - 1\right) + \frac{n}{r'} \right]$  or  $-\frac{n}{r'} - \left[ \left(1 - \frac{p}{r'}\right) + \frac{n(p-1)}{r'} \right] < \beta \leq -\frac{n}{r'}$ .

In other words, the available results using (1.13), (1.14) and (1.15) are no longer satisfactory to decide the one-weight boundedness (1.12). Therefore the question of deriving other criterion for (1.12), which includes the result of B. Muckenhoupt and R. Wheeden [Mu-Wh] described in (1.5), arises naturally. To solve such a problem is among the purposes in this work. Our main contribution here is also to bring a solution to the two-weight inequality (1.1).

Although a result about (1.1) for general weights  $u(\cdot)$  and  $v(\cdot)$  seems unknown until this paper, some authors in [Du], [Ho2], [Wa-Wh] have been already investigated the problem whenever the weight  $v(\cdot)$  is of the form  $(Su)(\cdot)$  where  $S$  is a suitable maximal operator. Most of the time, the operator  $S$  has a very complicated form since it is a composition of many other operators like the well-known Hardy littlewood maximal function

$$(Mf)(x) = \sup \left\{ |Q|^{-1} \int_Q |f(y)| dy; \ Q \text{ a cube containing } x \right\}.$$

Therefore, it is of interest to ask whether such an operator  $S$  can be replaced by the simple operator  $M$  when the weight  $u(\cdot)$  satisfies more reasonable assumptions. A positive answer to such a question will be brought here.

Our main result, which is Theorem 2.1, yields a sufficient condition ensuring the boundedness  $T : L^p(v(x)dx) \rightarrow L^p(u(x)dx)$ . The weights  $u(\cdot)$  and  $v(\cdot)$  we consider do not make use of any Muckenhoupt  $A_t$ -condition. In practical computations, the criteria we find seem to be easy to check since they are just expressed in term of behaviour of weights on annuli. Illustration over an explicit example, of pair of weights  $(u, v)$  with  $u \neq v$  for the two-weight inequality (1.1), is given in Proposition 2.2. The sufficient condition for (1.1) will be reduced, in Proposition 2.3, to the well-known Muckenhoupt type condition  $(u, v) \in A_p$  whenever the weights satisfy some growth assumptions. The boundedness  $T : L^p((Mu)(x)dx) \rightarrow L^p(u(x)dx)$  will be considered in Proposition 2.5.

Our results are stated in §2. Basic results useful for the proofs will be given in §3 and proved in §5. The proofs of the main results will be performed in §4.

## §2 Results

Throughout this paper it is assumed that

$$1 < p < \infty, \quad 1 \leq r < \infty, \quad p' = \frac{p}{p-1}$$

$r' = \frac{r}{r-1}$  for  $1 < r < \infty$  and  $r' = \infty$  if  $r = 1$ . The function  $\Omega(\cdot)$ , positively homogeneous of degree zero and with integral value zero on the unit sphere  $S^{n-1}$ , is supposed to be in  $L^r(S^{n-1})$ . That is

$$\|\Omega(\cdot)\|_{L^r(S^{n-1})} = \left( \int_{S^{n-1}} |\Omega(x')|^r d\sigma(x') \right)^{\frac{1}{r}} < \infty$$

where  $d\sigma(x')$  is the element area on  $S^{n-1}$  and  $x' = x|x|^{-1}$ . Moreover from now the unweighted boundedness

$$T = T_\Omega : L^p(dx) \rightarrow L^p(dx)$$

is assumed to be satisfied.

Recall that our main purpose in the present work is to derive sufficient conditions on weights  $u(\cdot)$  and  $v(\cdot)$  for which  $T$  is bounded from the weighted Lebesgue space  $L^p(v(x)dx)$  into  $L^p(u(x)dx)$ , or merely  $T : L^p(v(x)dx) \rightarrow L^p(u(x)dx)$ . That is for some constant  $C > 0$

$$(2.1) \quad \int_{\mathbb{R}^n} |(Tf)(x)|^p u(x) dx \leq (C \|\Omega(\cdot)\|_{L^r(S^{n-1})})^p \int_{\mathbb{R}^n} |f(x)|^p v(x) dx$$

for all  $f(\cdot) \in C_0^\infty(\mathbb{R}^n)$ . To simplify it is assumed that  $u(\cdot)$  and  $v(\cdot)$  are almost everywhere positive and finite functions.

Our main result is as follows.

**Theorem 2.1.** *The boundedness  $T : L^p(v(x)dx) \rightarrow L^p(u(x)dx)$  holds whenever for some  $\varepsilon, \varepsilon^*, A > 0$  the following three conditions are satisfied:*

$$(2.2) \quad \left( u(x) \right) \left( \sup_{4^{-1}|x| < |y| < 4|x|} \frac{1}{v(y)} \right) \leq A^p \quad \text{for a.e. } x;$$

$$(2.3) \quad \left( \sup_{R < |x| < 2R} u(x) \right)^{\frac{1}{p}} \left[ \sum_{j=1}^{\infty} 2^{-j(p'-1)[- \varepsilon + \mu(p, r, n)]} \times \right. \\ \left. \times \left( \sup_{2^{-1}(2-j)R < |y| < 2(2-j)R} v^{1-p'}(y) \right) \right]^{\frac{1}{p'}} \leq A \quad \text{for all } R > 0;$$

$$(2.3^*) \quad \left[ \sum_{j=1}^{\infty} 2^{-j(p-1)[- \varepsilon^* + \mu(p', r, n)]} \times \left( \sup_{2^{-1}(2^{-j}R) < |x| < 2(2^{-j}R)} u(x) \right) \right]^{\frac{1}{p}} \times \\ \times \left( \sup_{R < |y| < 2R} v^{1-p'}(y) \right)^{\frac{1}{p'}} \leq A \quad \text{for all } R > 0;$$

where  $\mu(p, r, n)$  and  $\mu(p', r, n)$  are given by

$$\mu(t, r, n) = \begin{cases} n(t-1) & \text{for } t \leq r \\ (t-1) + \frac{(n-1)t}{r'} & \text{for } r < t \end{cases}.$$

Precisely the constant  $C$  in (2.1) is of the form  $cA$  where  $c > 0$  depends only on  $n, p$  and  $r$ . Moreover the pointwise condition (2.2) can be replaced by

$$(2.2') \quad \left( \sup_{4^{-1}|x| < |y| < 4|x|} u(y) \right) \left( \frac{1}{v(x)} \right) \leq A^p \quad \text{for a.e. } x.$$

The case  $r = \infty$ , not treated in this paper, is included in Theorem 2 of [Ra2] where the boundedness  $T : L^p(v(x)dx) \rightarrow L^p(u(x)dx)$  is seen to be satisfied under both conditions (2.2) (or (2.2')) and

$$\left( \int_{2R < |x|} |x|^{-np} u(x) dx \right)^{\frac{1}{p}} \left( \int_{|y| < R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq A \quad \text{for all } R > 0$$

and

$$\left( \int_{|x| < R} u(x) dx \right)^{\frac{1}{p}} \left( \int_{2R < |y|} |y|^{-np'} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq A \quad \text{for all } R > 0.$$

Condition (2.3\*) is in somewhat a dual version of (2.3). Both of the two conditions are written in order to summarize six possibilities, that is if  $1 \leq r \leq 2$  then

$$(2.4) \quad \mu(p', r, n) = (p' - 1) + \frac{(n-1)p'}{r'} \quad \text{and} \quad \mu(p, r, n) = n(p-1) \quad \text{for } p \leq r \leq r'$$

$$(2.5) \quad \mu(p', r, n) = (p' - 1) + \frac{(n-1)p'}{r'} \quad \text{and} \\ \mu(p, r, n) = (p-1) + \frac{(n-1)p}{r'} \quad \text{for } r < p \leq r'$$

$$(2.6) \quad \mu(p', r, n) = n(p' - 1) \quad \text{and} \quad \mu(p, r, n) = (p-1) + \frac{(n-1)p}{r'} \quad \text{for } r \leq r' < p.$$

And if  $2 < r < \infty$  then

(2.7)

$$\mu(p', r, n) = (p' - 1) + \frac{(n-1)p'}{r'} \quad \text{and} \quad \mu(p, r, n) = n(p-1) \quad \text{for } p \leq r' < r$$

$$(2.8) \quad \mu(p', r, n) = n(p' - 1) \quad \text{and} \quad \mu(p, r, n) = n(p-1) \quad \text{for } r' < p \leq r$$

and

(2.9)

$$\mu(p', r, n) = n(p' - 1) \quad \text{and} \quad \mu(p, r, n) = (p-1) + \frac{(n-1)p}{r'} \quad \text{for } r' < r < p.$$

Theorem 2.1 includes the complete result, described in (1.6) to (1.11), about the boundedness (2.1) for power weights  $u(x) = v(x) = |x|^\beta$ . No interpolation with change of measures or any other arguments (see Theorem 6 of [Du], p. 874) combined with (1.13) and (1.14) are needed here to get the full results for power weights.

Therefore, even for the one-weight case  $u(\cdot) = v(\cdot)$ , our result is new and yields other boundedness criteria completing those described in (1.13), (1.14) and (1.15). And Theorem 2.1 is also a first solution to the two-weight problem (2.1).

To show the efficiency of conditions in Theorem 2.1 on deciding the boundedness (2.1) for  $u \neq v$ , we just state one example.

**Proposition 2.2.** *Define the weights*

$$u(x) = \begin{cases} |x|^{\alpha_1} & \text{for } |x| \leq 2^{-N_0} \\ |x|^{\alpha_2} & \text{for } |x| > 2^{-N_0} \end{cases} \quad \text{and} \quad v(x) = \begin{cases} |x|^{\beta_1} & \text{for } |x| \leq 2^{-N_0} \\ |x|^{\beta_2} & \text{for } |x| > 2^{-N_0} \end{cases}$$

where  $\alpha_1, \alpha_2, \beta_1$  and  $\beta_2$  are given real numbers and  $N_0$  is a nonnegative integer. Then the boundedness  $T : L^p(v(x)dx) \rightarrow L^p(u(x)dx)$  holds whenever  $\alpha_1, \alpha_2, \beta_1$  and  $\beta_2$  satisfy the following assumptions

$$(2.10) \quad \beta_1 \leq \alpha_1, \quad \alpha_2 \leq \beta_2$$

$$(2.11) \quad -(p-1)\mu(p', r, n) < \alpha_1, \quad \beta_1 < \mu(p, r, n)$$

and

$$(2.12) \quad -(p-1)\mu(p', r, n) < \alpha_2, \quad \beta_2 < \mu(p, r, n).$$

Here  $\mu(t, r, n)$  is defined as in Theorem 2.1.

Particularly the case  $u \neq v$  may be obtained for instance by taking

$$-(p-1)\mu(p', r, n) < \alpha_2 < \beta_2 \leq \beta_1 < \alpha_1 < \mu(p, r, n)$$

or

$$-(p-1)\mu(p', r, n) < \alpha_2 < \beta_1 \leq \beta_2 \leq \alpha_1 < \mu(p, r, n).$$

Of course for conditions (2.11) and (2.12) it is advisable to have in mind the six cases which are

- for  $1 \leq r \leq 2$  either  $p \leq r \leq r'$  or  $r < p \leq r'$  or  $r \leq r' < p$ ;

and

- for  $2 < r \leq \infty$  either  $p \leq r' < r$  or  $r' < p \leq r$  or  $r' < r \leq p$ .

Further examples of weights (as those stated in Corollary 7 of [Ra2]), which fall under the scope of criteria in Theorem 2.1, can be given but for shortness we will limit to Proposition 2.2.

Now the question, which arises, is about the connections of conditions (2.2), (2.3) and (2.3\*) with a more well-known Muckenhoupt type condition like

$$(2.13) \quad R^{-n} \left( \int_{|x| < R} u(x) dx \right)^{\frac{1}{p}} \left( \int_{|y| < R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq A \quad \text{for all } R > 0.$$

In contrast with the classical Muckenhoupt condition  $A_p$ , which is expressed in term of arbitrary balls (or cubes), only balls centered at the origin are involved in (2.13). So the advantage is that, for given weights  $u(\cdot)$  and  $v(\cdot)$ , explicit computations are in general easy to do particularly for radial weights. But the inconvenience with (2.13) is that, it is a too weak condition to derive boundedness (2.1) and more hypotheses on the weights will be needed.

An useful growth assumption for this purpose is:  $w(\cdot) \in \mathcal{H}$ , which means that for some integer  $N \geq 1$  and constant  $c > 0$

$$\sup_{4^{-1}R < |x| < 4R} w(x) \leq cR^{-n} \int_{2^{-N}R < |y| < 2^NR} w(y) dy \quad \text{for all } R > 0.$$

Many of usual weights satisfy this property. Particularly it is the case of any radial and monotone weight, for which  $N$  can be taken equal to 3. But the assumption  $w(\cdot) \in \mathcal{H}$  may be satisfied although  $w(\cdot)$  is not necessarily a monotone weight (see for instance Lemma 3 in [Ra2]).

Another growth condition, which will be helpful for the sequel, is the reverse doubling condition  $w(\cdot) \in RD_\rho$ ,  $\rho > 0$ . That is for some constant  $c > 0$

$$\int_{|y| < tR} w(y) dy \leq ct^{n\rho} \int_{|y| < R} w(y) dy \quad \text{for all } R > 0 \text{ and } 0 < t < 1.$$

Though the classical reverse doubling is about arbitrary cubes, here our condition  $w(\cdot) \in RD_\rho$  requires just the use of concentric balls centered at the origin. This fact will be valuable for explicit computations. Many weights satisfy the reverse doubling condition; in particular if  $w(\cdot) \in A_t$  then  $w(\cdot) \in RD_\rho$  for some  $\rho > 0$ , because  $w(\cdot)$  is a doubling weight.



**Proposition 2.3.** *Suppose that*

$$u(.) \in \mathcal{H} \cap RD_\tau \quad \text{and} \quad v^{1-p'}(.) \in \mathcal{H} \cap RD_\rho \quad \text{for some } \rho, \tau > 0.$$

*The Muckenhoupt type condition (2.13) becomes a sufficient condition for  $T : L^p(v(x)dx) \rightarrow L^p(u(x)dx)$  whenever one of the following cases is satisfied:*

**Case 1**  $1 \leq r \leq 2$ :

$$(2.14) \quad p \leq r \leq r' \quad \text{and} \quad 1 - \frac{1}{n} \left[ 1 + (n-1) \frac{p}{r'} \right] < \tau,$$

$$(2.15) \quad r < p \leq r' \quad \text{and} \quad 1 - \frac{1}{n} \left[ 1 + (n-1) \frac{p}{r'} \right] < \tau, \quad 1 - \frac{1}{n} \left[ 1 + (n-1) \frac{p'}{r'} \right] < \rho$$

$$(2.16) \quad r \leq r' < p \quad \text{and} \quad 1 - \frac{1}{n} \left[ 1 + (n-1) \frac{p'}{r'} \right] < \rho$$

**Case 2**  $2 < r < \infty$ :

$$(2.17) \quad p \leq r' < r \quad \text{and} \quad 1 - \frac{1}{n} \left[ 1 + (n-1) \frac{p}{r'} \right] < \tau,$$

$$(2.18) \quad r' < p \leq r$$

$$(2.19) \quad r' < r < p \quad \text{and} \quad 1 - \frac{1}{n} \left[ 1 + (n-1) \frac{p'}{r'} \right] < \rho.$$

In view of this result, the minimal hypotheses on  $u(.)$  and  $v^{1-p'}(.)$  occurs when  $2 \leq r < \infty$  with  $r' \leq p \leq r$ .

For  $u(x) = v(x) = |x|^\beta$ ,  $-n < \beta < n(p-1)$ , then  $u(.) \in \mathcal{H} \cap RD_\tau$  and  $v^{1-p'}(.) \in \mathcal{H} \cap RD_\rho$  with  $\tau = 1 + \frac{\beta}{n}$  and  $\rho = 1 - \frac{\beta(p'-1)}{n}$ . And easy computations show that assumptions (2.14) to (2.19) coincide with (1.6) to (1.11).

There are also two specific cases regarding to Proposition 2.3 which deserve to be noted.

**Corollary 2.4.** *Suppose that*

$$u(.) \in \mathcal{H} \cap RD_\tau \quad \text{and} \quad v^{1-p'}(.) \in \mathcal{H} \cap RD_\rho \quad \text{for some } \rho, \tau > 0.$$

*The Muckenhoupt type condition (2.13) becomes a sufficient condition for  $T : L^p(v(x)dx) \rightarrow L^p(u(x)dx)$  whenever one of the following cases is satisfied:*

$$(2.20) \quad \rho, \tau \geq 1,$$

$$(2.21) \quad n = 1.$$

So it appears that the one-dimensional case is a very different situation and the crux of matter arises when  $n \geq 2$ . Actually for  $n = 1$ , the operator  $T$  is essentially the well-known Hilbert transform  $(Hf)(x) = P.V. \int_{-\infty}^{\infty} \frac{1}{x-y} f(y) dy$ . Few pairs of weights  $(u, v)$  satisfy (2.20) and the main situations arise when  $0 < \rho, \tau < 1$ . Consequently Proposition 2.3 is really the interesting result, which in its turn leads to raise the problem about the exact values of  $\rho$  for a weight  $w(\cdot)$  satisfying a reverse doubling condition. A partial answer to such a question can be found in Proposition 8 of [Ra1].

Proposition 2.3 may be used as a tool to get  $T : L^p((Su)(x)dx) \rightarrow L^p(u(x)dx)$  for a simple operator  $S$ . To be coherent with results for the classical Calderón-Zygmund operator (see Proposition 11 in [Ra2]), the natural choice is that  $S = M$  where  $M$  is the Hardy-Littlewood maximal as recalled in the introduction.

**Proposition 2.5.** *Suppose that*

$$u(\cdot) \in \mathcal{H} \cap RD_\tau \quad \text{for some } \tau > 0.$$

*Then there is a constant  $C > 0$  such that*

$$(2.22) \quad \int_{\mathbb{R}^n} |(Tf)(x)|^p u(x) dx \leq (C \|\Omega(\cdot)\|_{L^r(S^{n-1})})^p \int_{\mathbb{R}^n} |f(x)|^p (Mu)(x) dx \quad \text{for all } f(\cdot) \in C_0^\infty(\mathbb{R}^n)$$

*whenever one of the following assumptions is satisfied:*

$$(2.23) \quad n = 1,$$

$$(2.24) \quad 1 \leq r \leq 2, \quad p \leq r \leq r', \quad 1 - \frac{1}{n} \left[ 1 + (n-1) \frac{p}{r'} \right] < \tau,$$

$$(2.25) \quad 2 < r < \infty, \quad p \leq r' < r, \quad 1 - \frac{1}{n} \left[ 1 + (n-1) \frac{p}{r'} \right] < \tau,$$

$$(2.26) \quad 2 \leq r < \infty, \quad r' \leq p \leq r.$$

The constant  $C > 0$  in (2.22) depends only on  $n, p, r$  and on the fact that  $u(\cdot) \in \mathcal{H} \cap RD_\rho$ .

### §3 Auxiliary Results

This section is devoted to the statements of some auxiliary results needed for the proofs of our main results.

One of the keys about Theorem 2.1, also interesting for its own sake, is the following sort of Hardy inequality.

**Theorem 3.1.** *Let  $K(x, y)$  be a nonnegative kernel such that for some constant  $\mathcal{K} > 0$*

$$(3.1) \quad \int_{2^{-(j+1)}|x| < |z| < 2^{-j}|x|} K^r(x, z) dz \leq (2^{-j}|x|^n) \mathcal{K}^r \quad \text{for all integers } j \geq 1$$

and

$$(3.2) \quad \int_{2^k|y| < |x| < 2^{k+1}|y|} K^r(x, y) dx \leq (2^k|y|^n) \mathcal{K}^r \quad \text{for all integers } k \geq 1.$$

Let  $1 < t < \infty$ ,  $t' = \frac{t}{t-1}$  and  $\bar{u}(\cdot)$ ,  $\bar{v}(\cdot)$  be weights such that for some  $\varepsilon, A > 0$

$$(3.3) \quad \left( \sup_{R < |x| < 2R} \bar{u}(x) \right)^{\frac{1}{t}} \left[ \sum_{j=1}^{\infty} 2^{-j(t'-1)[- \varepsilon + \mu(t, r, n)]} \times \right. \\ \left. \times \left( \sup_{2^{-1}(2^{-j}R) < |y| < 2(2^{-j}R)} \bar{v}^{(1-t')}(y) \right) \right]^{\frac{1}{t'}} \leq A \quad \text{for all } R > 0,$$

where  $\mu(t, r, n)$  is defined as in Theorem 2.1. Then for some constant  $c > 0$

$$(3.4) \quad \int_{x \in \mathbb{R}^n} \left[ \int_{|y| < 2^{-1}|x|} K(x, y) g(y) dy \right]^t |x|^{-nt} \bar{u}(x) dx \leq (cA\mathcal{K})^t \int_{y \in \mathbb{R}^n} g^t(y) \bar{v}(y) dy$$

for all functions  $g(\cdot) \geq 0$ .

This result will be proved in §5 by using

**Lemma 3.2.** *Let  $K(x, y)$  be a nonnegative kernel and  $1 < t < \infty$  with  $t' = \frac{t}{t-1}$ . Assume that for some constant  $A > 0$  and function  $\varphi(\cdot)$ :*

$$(3.5) \quad \varphi(y) \left( \int_{2|y| < |x|} K^{mt}(x, y) \Theta^{t-1}(x) |x|^{-nt} \bar{u}(x) dx \right) \leq A^t \quad \text{for a.e. } y \in \mathbb{R}^n$$

where

$$(3.6) \quad \Theta(x) = \int_{|z| < 2^{-1}|x|} K^{(1-m)t'}(x, z) \varphi^{1-t'}(z) \bar{v}^{(1-t')}(z) dz \quad \text{with } m = \min(1, \frac{r}{t}).$$

Then

$$(3.7) \quad \int_{x \in \mathbb{R}^n} \left[ \int_{|y| < 2^{-1}|x|} K(x, y) g(y) dy \right]^t |x|^{-nt} \bar{u}(x) dx \leq A^t \int_{y \in \mathbb{R}^n} g^t(y) \bar{v}(y) dy$$

for all functions  $g(\cdot) \geq 0$ .

In order to apply Theorem 3.1, it is helpful to have

**Lemma 3.3.** *There is a constant  $c > 0$ , which only depends on the dimension  $n$ , such that*

$$(3.8) \quad \int_{2^{-(j+1)}|x| < |z| < 2^{-j}|x|} |\Omega(x - z)|^r dz \\ + \int_{2^{-(j+1)}|x| < |z| < 2^{-j}|x|} |\Omega(z - x)|^r dz \leq c(2^{-j}|x|^n) \|\Omega(\cdot)\|_{L^r(S^{n-1})}^r$$

for all integers  $j \geq 1$  and

$$(3.9) \quad \int_{2^k|y| < |x| < 2^{k+1}|y|} |\Omega(x - y)|^r dx \\ + \int_{2^k|y| < |x| < 2^{k+1}|y|} |\Omega(y - x)|^r dx \leq c(2^k|y|^n) \|\Omega(\cdot)\|_{L^r(S^{n-1})}^r$$

for all integers  $k \geq 1$ .

This last Lemma and Theorem 3.1 can be used to derive

**Proposition 3.4.** *Assume that condition (2.3) (in Theorem 2.1) is satisfied. Then for some constant  $c > 0$*

$$(3.10) \quad \int_{x \in \mathbb{R}^n} \left[ \int_{|y| < 2^{-1}|x|} |\Omega(x - y)| g(y) dy \right]^p |x|^{-np} u(x) dx \\ \leq (cA \|\Omega(\cdot)\|_{L^r(S^{n-1})})^p \int_{y \in \mathbb{R}^n} g^p(y) v(y) dy$$

for all functions  $g(\cdot) \geq 0$ .

Once again Theorem 3.1 and Lemma 3.3 are the main points for

**Proposition 3.5.** *Assume that condition (2.3\*) (in Theorem 2.1) is satisfied. Then for some constant  $c > 0$*

$$(3.11) \quad \int_{x \in \mathbb{R}^n} \left[ \int_{2|x| < |y|} |\Omega(x - y)| |y|^{-n} g(y) dy \right]^p u(x) dx \\ \leq (cA \|\Omega(\cdot)\|_{L^r(S^{n-1})})^p \int_{y \in \mathbb{R}^n} g^p(y) v(y) dy$$

for all functions  $g(\cdot) \geq 0$ .

For the proof of Theorem 3.1, the following estimates will be useful

**Lemma 3.6.** *Let  $K(x, y)$  be a nonnegative kernel satisfying assumptions (3.1)*

*and (3.2). Define  $m = \min(1, \frac{r}{t})$  and  $\tau(t, r, n) = \begin{cases} 0 & \text{for } t \leq r \\ (n-1)\left(\frac{t-r}{t-1}\right)\frac{1}{r} & \text{for } r < t \end{cases}$ .*

*Then for some constant  $c > 0$*

$$(3.12) \quad \int_{2^{-(j+1)}|x| < |z| < 2^{-j}|x|} K^{(1-m)t'}(x, z) dz \\ \leq c(2^{-j[n-\tau(t, r, n)]}|x|^n) \mathcal{K}^{(1-m)t'} \quad \text{for all integers } j \geq 1$$

*and*

$$(3.13) \quad \int_{2^k|y| < |x| < 2^{k+1}|y|} K^{mt}(x, y) dx \leq c(2^k|y|)^n \mathcal{K}^{mt} \quad \text{for all integers } k \geq 1.$$

Finally Propositions 2.3 and 2.5 will be derived from

**Lemma 3.7.** *Let  $1 < t < \infty$ ,  $t' = \frac{t}{t-1}$ . And let  $\bar{u}(\cdot)$  and  $\bar{v}(\cdot)$  be weights such that  $\bar{u}(\cdot) \in \mathcal{H}$  and  $\bar{v}^{(1-t')}(\cdot) \in \mathcal{H} \cap RD_\rho$  for some  $\rho > 0$ .*

*For  $t \leq r$ , then condition (3.3) is implied by*

$$(3.14) \quad R^{-n} \left( \int_{|x| < R} \bar{u}(x) dx \right)^{\frac{1}{t}} \left( \int_{|y| < R} \bar{v}^{(1-t')}(y) dy \right)^{\frac{1}{t'}} \leq A \quad \text{for all } R > 0.$$

*And for  $r < t$  the same implication holds whenever*

$$(3.15) \quad 1 - \frac{1}{n} \left[ 1 + \frac{(n-1)t'}{r'} \right] < \rho.$$

## §4 Proofs of Main Results

### Proof of Theorem 2.1

To prove the boundedness  $T : L^p(v(x)dx) \rightarrow L^p(u(x)dx)$  let us consider a function  $f(\cdot) \in C_0^\infty(\mathbb{R}^n)$ . Then for some constant  $c = c(p) > 0$

$$\int_{\mathbb{R}^n} |(Tf)(x)|^p u(x) dx \leq c(\mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3)$$

where

$$\mathcal{S}_1 = \sum_k \int_{E_k} |(Tf \mathbb{I}_{L_k})(x)|^p u(x) dx \\ \mathcal{S}_2 = \sum_k \int_{E_k} |(Tf \mathbb{I}_{M_k})(x)|^p u(x) dx$$

$$\mathcal{S}_3 = \sum_k \int_{E_k} |(Tf \mathbb{I}_{R_k})(x)|^p u(x) dx$$

with

$$E_k = \{x; 2^k < |x| \leq 2^{k+1}\}, \quad M_k = \{x; 2^{k-1} \leq |x| \leq 2^{k+2}\}$$

and

$$L_k = \{x; |x| < 2^{k-1}\}, \quad R_k = \{x; 2^{k+2} < |x|\}.$$

Here  $\mathbb{I}_E$  is used to denote the characteristic function of the measurable set  $E$ .

### Estimate of $\mathcal{S}_1$

The function  $(f \mathbb{I}_{L_k})(\cdot) \in L_c^\infty(\mathbb{R}^n)$  has a support contained in the ball  $B(0, 2^{k-1}) = \{y; |y| < 2^{k-1}\}$ . Since  $T : L^p(\mathbb{R}^n, dx) \rightarrow L^p(\mathbb{R}^n, dx)$  then  $(Tf \mathbb{I}_{L_k})(x)$  has a sense for each  $x \in E_k$  and

$$\begin{aligned} & \left| (Tf \mathbb{I}_{L_k})(x) \right| \\ &= \left| \int_{y \in L_k} \Omega(x-y) |x-y|^{-n} f(y) dy \right| \\ & \quad \text{since } x \notin B(0, 2^{k-1}) \\ &\leq \int_{|y| < 2^{-1}|x|} |\Omega(x-y)| |x-y|^{-n} |f(y)| dy \\ &\leq 2^n |x|^{-n} \int_{|y| < 2^{-1}|x|} |\Omega(x-y)| |f(y)| dy \quad \text{since } \frac{1}{2}|x| < |x-y| < \frac{3}{2}|x|. \end{aligned}$$

Using this last inequality, condition (2.3) and Proposition 3.4 then

$$\begin{aligned} \mathcal{S}_1 &\leq 2^{np} \int_{\mathbb{R}^n} \left[ \int_{|y| < 2^{-1}|x|} |\Omega(x-y)| |f(y)| dy \right]^p |x|^{-np} u(x) dx \\ &\leq (c_1 A \|\Omega(\cdot)\|_{L^r(S^{n-1})})^p \int_{\mathbb{R}^n} |f(y)|^p v(y) dy \quad \text{see (3.10)}. \end{aligned}$$

### Estimate of $\mathcal{S}_3$

As above, for each  $x \in E_k$ ,  $x$  does not belong to the support of the function  $(f \mathbb{I}_{R_k})(\cdot)$ . Then

$$\begin{aligned} & \left| (Tf \mathbb{I}_{R_k})(x) \right| \leq \int_{2|x| < |y|} |\Omega(x-y)| |x-y|^{-n} |f(y)| dy \\ &\leq 2^n \int_{2|x| < |y|} |\Omega(x-y)| |f(y)| |y|^{-n} dy \quad \text{since } \frac{1}{2}|y| < |x-y| < 2|y|. \end{aligned}$$

This last inequality, condition (2.3\*) and Proposition 3.5 lead to

$$\begin{aligned} \mathcal{S}_3 &\leq 2^{np} \int_{\mathbb{R}^n} \left[ \int_{2|x| < |y|} |\Omega(x-y)| |f(y)| |y|^{-n} dy \right]^p u(x) dx \\ &\leq (c_1 A \|\Omega(\cdot)\|_{L^r(S^{n-1})})^p \int_{\mathbb{R}^n} |f(y)|^p v(y) dy \quad \text{see (3.11).} \end{aligned}$$

### Estimate of $\mathcal{S}_2$

Suppose, for instance, that condition (2.2') is satisfied. Then

$$\begin{aligned} \mathcal{S}_2 &= \sum_k \int_{E_k} \left| (Tf \mathbb{I}_{M_k})(x) \right|^p u(x) dx \\ &\leq \sum_k \left[ \sup_{z \in E_k} u(z) \right] \int_{E_k} \left| (Tf \mathbb{I}_{M_k})(x) \right|^p dx \\ &\leq (c_2 \|\Omega(\cdot)\|_{L^r(S^{n-1})})^p \sum_k \int_{M_k} |f(x)|^p \left[ \sup_{z \in E_k} u(z) \right] dx \\ &\quad \text{since } T : L^p(\mathbb{R}^n, dx) \rightarrow L^p(\mathbb{R}^n, dx) \\ &\leq (c_2 \|\Omega(\cdot)\|_{L^r(S^{n-1})})^p \sum_k \int_{M_k} |f(x)|^p \left[ \sup_{\frac{1}{4}|x| < |z| < 4|x|} u(z) \right] dx \\ &\leq (c_2 A \|\Omega(\cdot)\|_{L^r(S^{n-1})})^p \sum_k \int_{M_k} |f(x)|^p v(x) dx \quad \text{by condition (2.2')} \\ &= 3(c_2 A \|\Omega(\cdot)\|_{L^r(S^{n-1})})^p \int_{\mathbb{R}^n} |f(x)|^p v(x) dx. \end{aligned}$$

If instead of (2.2') condition (2.2) is supposed to be satisfied then observe that

$$\sup_{z \in E_k} \left[ \frac{1}{\sup_{\frac{1}{4}|z| < |y| < 4|z|} \frac{1}{v(y)}} \right] \leq v(x) \quad \text{for all } x \in M_k.$$

Indeed for  $x \in M_k$  and  $z \in E_k$  then  $\frac{1}{4}|z| < 2^{k-1} < |x| < 2^{k+2} < 4|z|$ . And consequently  $1 = \frac{1}{v(x)} \times v(x) < \left( \sup_{\{\frac{1}{4}|z| < |y| < 4|z|\}} \frac{1}{v(y)} \right) \times v(x)$  or

$\frac{1}{\sup_{\frac{1}{4}|z| < |y| < 4|z|} \frac{1}{v(y)}} \leq v(x)$  which implies the desired estimate. Condition (2.2) and the previous observation lead to

$$\begin{aligned} \mathcal{S}_2 &\leq (c_2 \|\Omega(\cdot)\|_{L^r(S^{n-1})})^p \sum_k \int_{M_k} |f(x)|^p \left[ \sup_{z \in E_k} u(z) \right] dx \quad \text{see above} \\ &\leq (c_2 A \|\Omega(\cdot)\|_{L^r(S^{n-1})})^p \sum_k \int_{M_k} |f(x)|^p \left[ \sup_{z \in E_k} \frac{1}{\sup_{\frac{1}{4}|z| < |y| < 4|z|} \frac{1}{v(y)}} \right] dx \end{aligned}$$

$$\begin{aligned}
&\leq (c_2 A \|\Omega(\cdot)\|_{L^r(S^{n-1})})^p \sum_k \int_{M_k} |f(x)|^p v(x) dx \\
&= 3(c_2 A \|\Omega(\cdot)\|_{L^r(S^{n-1})})^p \int_{\mathbb{R}^n} |f(x)|^p v(x) dx.
\end{aligned}$$

### Proof of Proposition 2.2

By Theorem 2.1 to get  $T : L^p(v(x)dx) \rightarrow L^p(u(x)dx)$ , our task is to check conditions (2.2), (2.3) and (2.3\*).

#### Condition (2.2)

The requirement for this condition means that

$$\mathcal{P}(x) = u(x) \left( \sup_{4^{-1}|x| < |y| < 4|x|} \frac{1}{v(y)} \right)$$

is bounded by a constant which does not depend on  $x$ .

When  $|x|$  is small i.e.  $|x| < 4^{-1}2^{-N_0}$  then

$$\begin{aligned}
\mathcal{P}(x) &\approx |x|^{\alpha_1} \times |x|^{-\beta_1} = |x|^{\alpha_1 - \beta_1} < (4^{-1}2^{-N_0})^{\alpha_1 - \beta_1} \\
&\text{since by (2.10): } \alpha_1 - \beta_1 \geq 0.
\end{aligned}$$

And for large  $|x|$  s i.e.  $42^{-N_0} < |x|$  then

$$\begin{aligned}
\mathcal{P}(x) &\approx |x|^{\alpha_2} \times |x|^{-\beta_2} = |x|^{\alpha_2 - \beta_2} < (42^{-N_0})^{\alpha_2 - \beta_2} \\
&\text{since by (2.10): } \alpha_2 - \beta_2 \leq 0.
\end{aligned}$$

And for  $|x| \approx 1$  in the sense that  $4^{-1}2^{-N_0} \leq |x| \leq 42^{-N_0}$  it is obvious that  $\mathcal{P}(x) < C$  for some constant  $C > 0$  which depends only on  $N_0$ .

#### Condition (2.3)

We have to prove that

$$\begin{aligned}
\mathcal{A}(R) &= \left( \sup_{R < |x| < 2R} u(x) \right)^{\frac{1}{p}} \left[ \sum_{j=1}^{\infty} 2^{-j(p'-1)[- \varepsilon + \mu(p, r, n)]} \times \right. \\
&\quad \left. \times \left( \sup_{2^{-1}(2^{-j}R) < |y| < 2(2^{-j}R)} v^{1-p'}(y) \right) \right]^{\frac{1}{p'}}
\end{aligned}$$

is bounded by a constant which depends only on  $N_0$  for all  $R > 0$  and with  $\varepsilon > 0$  chosen such that

$$\varepsilon < \min\{\mu(p, r, n), \mu(p, r, n) - \beta_1, \mu(p, r, n) - \beta_2\}.$$

When  $R$  is small i.e.  $0 < R < 2^{-(1+N_0)}$  then

$$\mathcal{A}(R) \approx R^{\frac{\alpha_1}{p}} \left[ \sum_{j=1}^{\infty} 2^{-j(p'-1)[- \varepsilon + \mu(p, r, n)]} (2^{-j}R)^{(1-p')\beta_1} \right]^{\frac{1}{p'}}$$



$$\begin{aligned}
&= R^{(\alpha_1 - \beta_1) \frac{1}{p}} \left[ \sum_{j=1}^{\infty} 2^{-j(p'-1)[- \varepsilon + \mu(p, r, n) - \beta_1]} \right]^{\frac{1}{p'}} \\
&\leq C(p, r, n, \varepsilon, \beta_1) 2^{-(1+N_0)(\alpha_1 - \beta_1) \frac{1}{p}} \\
&\quad \text{since } \alpha_1 - \beta_1 \geq 0 \text{ and } -\varepsilon + \mu(p, r, n) - \beta_1 > 0.
\end{aligned}$$

For the case when  $R$  is large i.e.  $82^{-N_0} < R$  observe that

$$\begin{aligned}
\mathcal{A}(R) &\approx R^{\frac{\alpha_2}{p}} \left[ \sum_{j=1}^{\infty} 2^{-j(p'-1)[- \varepsilon + \mu(p, r, n)]} \left( \sup_{2^{-1}(2^{-j}R) < |y| < 2(2^{-j}R)} v^{1-p'}(y) \right) \right]^{\frac{1}{p'}} \\
&\approx \mathcal{A}_1(R) + \mathcal{A}_2(R) + \mathcal{A}_3(R)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}_1(R) &= R^{\frac{\alpha_2}{p}} \left[ \sum_{1 \leq j < \frac{\ln(2^{N_0-1}R)}{\ln 2}} 2^{-j(p'-1)[- \varepsilon + \mu(p, r, n)]} \times \right. \\
&\quad \left. \times \left( \sup_{2^{-1}(2^{-j}R) < |y| < 2(2^{-j}R)} v^{1-p'}(y) \right) \right]^{\frac{1}{p'}}
\end{aligned}$$

$$\begin{aligned}
&\mathcal{A}_2(R) \\
&= R^{\frac{\alpha_2}{p}} \left[ \sum_{\frac{\ln(2^{N_0+1}R)}{\ln 2} < j} 2^{-j(p'-1)[- \varepsilon + \mu(p, r, n)]} \left( \sup_{2^{-1}(2^{-j}R) < |y| < 2(2^{-j}R)} v^{1-p'}(y) \right) \right]^{\frac{1}{p'}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_3(R) &= R^{\frac{\alpha_2}{p}} \left[ \sum_{\frac{\ln(2^{N_0-1}R)}{\ln 2} \leq j \leq \frac{\ln(2^{N_0+1}R)}{\ln 2}} 2^{-j(p'-1)[- \varepsilon + \mu(p, r, n)]} \times \right. \\
&\quad \left. \times \left( \sup_{2^{-1}(2^{-j}R) < |y| < 2(2^{-j}R)} v^{1-p'}(y) \right) \right]^{\frac{1}{p'}}
\end{aligned}$$

The estimates for  $\mathcal{A}_1(R)$  are as follows

$$\begin{aligned}
\mathcal{A}_1(R) &\approx R^{\frac{\alpha_2}{p}} \left[ \sum_{1 \leq j < \frac{\ln(2^{N_0-1}R)}{\ln 2}} 2^{-j(p'-1)[- \varepsilon + \mu(p, r, n)]} (2^{-j}R)^{(1-p')\beta_2} \right]^{\frac{1}{p'}} \\
&= R^{(\alpha_2 - \beta_2) \frac{1}{p}} \left[ \sum_{j=1}^{\infty} 2^{-j(p'-1)[- \varepsilon + \mu(p, r, n) - \beta_2]} \right]^{\frac{1}{p'}}
\end{aligned}$$

$$\leq C(p, r, n, \varepsilon, \alpha_2, \beta_2) 2^{(3-N_0)(\alpha_2-\beta_2)\frac{1}{p}}$$

since  $\alpha_2 - \beta_2 \leq 0$  and  $-\varepsilon + \mu(p, r, n) - \beta_2 > 0$ .

And  $\mathcal{A}_2(R)$  can be handled as follows

$$\begin{aligned} \mathcal{A}_2(R) &\approx R^{\frac{\alpha_2}{p}} \left[ \sum_{\frac{\ln(2^{N_0+1}R)}{\ln 2} < j} 2^{-j(p'-1)[- \varepsilon + \mu(p, r, n)]} (2^{-j}R)^{(1-p')\beta_1} \right]^{\frac{1}{p'}} \\ &= R^{(\alpha_2-\beta_1)\frac{1}{p}} \left[ \sum_{\frac{\ln(2^{N_0+1}R)}{\ln 2} < j} 2^{-j(p'-1)[- \varepsilon + \mu(p, r, n) - \beta_1]} \right]^{\frac{1}{p'}} \\ &\leq c_1(N_0) R^{[(\alpha_2-\beta_1)+\varepsilon-\mu(p, r, n)+\beta_1]\frac{1}{p}} \\ &= c_1(N_0) R^{[\varepsilon+\alpha_2-\mu(p, r, n)]\frac{1}{p}} \\ &\leq c_2(N_0) 2^{(3-N_0)(\varepsilon+\alpha_2-\mu(p, r, n))\frac{1}{p}} \\ &\quad \text{since } \varepsilon + \alpha_2 - \mu(p, r, n) \leq \varepsilon + \beta_2 - \mu(p, r, n) < 0. \end{aligned}$$

And  $\mathcal{A}_3(R)$  can be also bounded since

$$\begin{aligned} \mathcal{A}_3(R) &\leq c_3(N_0) R^{\frac{\alpha_2}{p}} \left[ \sum_{\frac{\ln(2^{N_0-1}R)}{\ln 2} \leq j \leq \frac{\ln(2^{N_0+1}R)}{\ln 2}} 2^{-j(p'-1)[- \varepsilon + \mu(p, r, n)]} \right]^{\frac{1}{p'}} \\ &\leq c_4(N_0) R^{\frac{\alpha_2}{p}} R^{-[- \varepsilon + \mu(p, r, n)]\frac{1}{p}} \\ &= c_4(N_0) R^{[\alpha_2+\varepsilon-\mu(p, r, n)]\frac{1}{p}} \\ &\leq c_5(N_0) 2^{(3-N_0)(\alpha_2+\varepsilon-\mu(p, r, n))\frac{1}{p}} \quad \text{because } \alpha_2 + \varepsilon - \mu(p, r, n) < 0. \end{aligned}$$

Finally the case  $R \approx 1$ , in the sense that  $2^{(-1-N_0)} \leq R \leq 2^{3-N_0}$ , can be easily treated since

$$\begin{aligned} &\left( \sup_{R < |x| < 2R} u(x) \right)^{\frac{1}{p}} \left[ \sum_{j=1}^3 2^{-j(p'-1)[- \varepsilon + \mu(p, r, n)]} \left( \sup_{2^{-1}(2^{-j}R) < |y| < 2(2^{-j}R)} v^{1-p'}(y) \right) \right]^{\frac{1}{p'}} \\ &\leq c_6(N_0, p, r, n) \end{aligned}$$

and

$$\begin{aligned} &\left( \sup_{R < |x| < 2R} u(x) \right)^{\frac{1}{p}} \left[ \sum_{j=4}^{\infty} 2^{-j(p'-1)[- \varepsilon + \mu(p, r, n)]} \left( \sup_{2^{-1}(2^{-j}R) < |y| < 2(2^{-j}R)} v^{1-p'}(y) \right) \right]^{\frac{1}{p'}} \\ &\leq c_7(N_0, \beta_1, p) \left[ \sum_{j=1}^{\infty} 2^{-j(p'-1)[- \varepsilon + \mu(p, r, n) - \beta_1]} \right]^{\frac{1}{p'}} \end{aligned}$$

$$\leq c_8(N_0, \beta_1, p, r) \quad \text{because } -\varepsilon + \mu(p, r, n) - \beta_1 > 0.$$

**Condition (2.3\*)**

The task remains to prove that

$$\begin{aligned} \mathcal{A}^*(R) = & \left( \sup_{R < |x| < 2R} u_1(x) \right)^{\frac{1}{p_1}} \left[ \sum_{j=1}^{\infty} 2^{-j(p'_1-1)[- \varepsilon^* + \mu(p_1, r, n)]} \times \right. \\ & \left. \times \left( \sup_{2^{-1}(2^{-j}R) < |y| < 2(2^{-j}R)} v_1^{1-p'_1}(y) \right) \right]^{\frac{1}{p_1}} \end{aligned}$$

is bounded by a constant which depends only on  $N_0$  for all  $R > 0$ . Here  $p_1 = p'$ ,  $p'_1 = p$ ,

$$\begin{aligned} u_1(x) &= v^{1-p'}(x) \\ &= \begin{cases} |x|^{\alpha_1^*} & \text{for } |x| \leq 2^{-N_0} \\ |x|^{\alpha_2^*} & \text{for } |x| > 2^{-N_0}, \end{cases} \quad \alpha_1^* = (1-p')\beta_1 \text{ and } \alpha_2^* = (1-p')\beta_2; \end{aligned}$$

$$\begin{aligned} v_1(x) &= u^{\frac{1}{1-p}}(x) \\ &= \begin{cases} |x|^{\beta_1^*} & \text{for } |x| \leq 2^{-N_0} \\ |x|^{\beta_2^*} & \text{for } |x| > 2^{-N_0} \end{cases} \quad \beta_1^* = \frac{1}{1-p}\alpha_1 \text{ and } \beta_2^* = \frac{1}{1-p}\alpha_2. \end{aligned}$$

The arguments used in condition (2.3) show that  $\mathcal{A}^*(R)$  can be bounded by a fixed constant (essentially depending on  $N_0$ ), for all  $R > 0$ , whenever

$$\beta_1^* \leq \alpha_1^*, \quad \alpha_2^* \leq \beta_2^*, \quad \beta_1^* < \mu(p_1, r, n) \quad \text{and} \quad \beta_2^* < \mu(p_1, r, n).$$

And these assumptions mean respectively

$$\beta_1 \leq \alpha_1, \quad \alpha_2 \leq \beta_2, \quad -(p-1)\mu(p', r, n) < \alpha_1 \quad \text{and} \quad -(p-1)\mu(p', r, n) < \alpha_2.$$

Therefore condition (2.3\*) is satisfied.

**Proof of Proposition 2.3**

To get  $T : L^p(v(x)dx) \rightarrow L^p(u(x)dx)$ , by Theorem 2.1, it remains to check conditions (2.2), (2.3) and (2.3\*).

With  $u(\cdot)$ ,  $v^{1-p'}(\cdot) \in \mathcal{H}$ , it is clear that (2.2) is implied by the Muckenhoupt condition (2.13). And the point key for the remaining two conditions is essentially Lemma 3.7.

For example, if  $p \leq r$  and  $r \leq p'$  (the cases described in (2.14) and (2.17)) then (2.3) is implied by (2.13) whenever  $u(\cdot) \in \mathcal{H}$  and  $v^{1-p'}(\cdot) \in \mathcal{H} \cap RD_\rho$ .

And (2.3\*) is implied by (2.13) whenever  $v^{1-p'}(.) \in \mathcal{H}$ ,  $u(.) \in \mathcal{H} \cap RD_\tau$  with  $1 - \frac{1}{n} \left[ 1 + \frac{(n-1)p}{r'} \right] < \tau$ . The other cases  $r \leq p$ ,  $p' \leq r$  or  $r \leq p$ ,  $r \leq p'$  and  $p \leq r$ ,  $p' \leq r$  can be treated similarly without any supplementary difficulty.

**Proof of Proposition 2.5**

It is known from Proposition 11 in [Ra2] that the pair  $(u(.), v(.)) = (Mu(.))$  satisfies the Muckenhoupt condition (2.13) under the assumption  $u(.) \in \mathcal{H} \cap RD_\tau$  for some  $\tau > 0$ .

Conclusion in (2.23) follows from (2.21) in Corollary 2.4. And the results stated in (2.24), (2.25) and (2.26) respectively will appear immediately from (2.14), (2.17) and (2.18).

## §5 Proofs of Auxiliary Results

**Proof of Lemma 3.2**

To derive inequality (3.7) let us define  $m$  and  $\Theta(.)$  as in (3.6). Observe that by the Hölder inequality

$$\int_{|y| < 2^{-1}|x|} K(x, y)g(y)dy \leq \left( \int_{|y| < 2^{-1}|x|} K^{mt}(x, y)g^t(y)\bar{v}(y)\varphi(y)dy \right)^{\frac{1}{t}} \Theta^{\frac{1}{t'}}(x).$$

Then the conclusion follows immediately after using condition (3.5) since

$$\begin{aligned} & \int_{x \in \mathbb{R}^n} \left[ \int_{|y| < 2^{-1}|x|} K(x, y)g(y)dy \right]^t |x|^{-nt}\bar{u}(x)dx \\ & \leq \int_{x \in \mathbb{R}^n} \left[ \int_{|y| < 2^{-1}|x|} K^{mt}(x, y)g^t(y)\bar{v}(y)\varphi(y)dy \right] \Theta^{t-1}(x) |x|^{-nt}\bar{u}(x)dx \\ & = \int_{y \in \mathbb{R}^n} g^t(y)\bar{v}(y)\varphi(y) \left[ \int_{2|y| < |x|} K^{mt}(x, y)\Theta^{t-1}(x) |x|^{-nt}\bar{u}(x)dx \right] dy \\ & \leq A^t \int_{y \in \mathbb{R}^n} g^t(y)\bar{v}(y)dy \quad \text{by condition (3.5).} \end{aligned}$$

**Proof of Lemma 3.3**

The sequence of computations which leads to the one part of inequality (3.8) is the following

$$\begin{aligned} & \int_{2^{-(j+1)}|x| < |z| < 2^{-j}|x|} |\Omega(x-z)|^r dz = \int_{2^{-(j+1)}|x| < |x-y| < 2^{-j}|x|} |\Omega(y)|^r dy \\ & \leq \int_{(1-2^{-j})|x| < |y| < (1+2^{-j})|x|} |\Omega(y)|^r dy \quad \text{recall that } j \geq 1 \\ & \approx \int_{(1-2^{-j})|x|}^{(1+2^{-j})|x|} \rho^{n-1} \left( \int_{S^{n-1}} |\Omega(\rho\sigma)|^r d\sigma \right) d\rho \quad \text{by using polar coordinates} \end{aligned}$$

$$\approx \left( \int_{S^{n-1}} |\Omega(\sigma)|^r d\sigma \right) \int_{(1-2^{-j})|x|}^{(1+2^{-j})|x|} \rho^{n-1} d\rho$$

by the homogeneity of degree zero of  $\Omega(\cdot)$

$$\begin{aligned} &\approx \|\Omega(\cdot)\|_{L^r(S^{n-1})}^r \left[ (1+2^{-j})^n - (1-2^{-j})^n \right] |x|^n \\ &\leq c 2^{-j} |x|^n \|\Omega(\cdot)\|_{L^r(S^{n-1})}^r \quad \text{where } c > 0 \text{ depends only on } n. \end{aligned}$$

The second part of (3.8) can be seen similarly as above.

The first part of inequality (3.9) follows also from analogue arguments since

$$\begin{aligned} &\int_{2^k|y| < |x| < 2^{k+1}|y|} |\Omega(x-y)|^r dx = \int_{2^k|y| < |y+z| < 2^{k+1}|y|} |\Omega(z)|^r dz \\ &\leq \int_{|z| < 2^{k+3}|y|} |\Omega(z)|^r dz \approx \int_0^{2^{k+3}|y|} \rho^{n-1} \left( \int_{S^{n-1}} |\Omega(\rho\sigma)|^r d\sigma \right) d\rho \\ &\leq c(2^k|y|)^n \|\Omega(\cdot)\|_{L^r(S^{n-1})}^r \quad \text{where } c > 0 \text{ depends only on } n. \end{aligned}$$

Details for the second part of (3.9) are left again to the readers.

#### Proof of Proposition 3.4

With the notations used in Theorem 3.1 then  $t = p$ ,  $K(x, y) = |\Omega(x - y)|$ ,  $\mathcal{K}^r = c\|\Omega(\cdot)\|_{L^r(S^{n-1})}^r$ ,  $\bar{u}(\cdot) = u(\cdot)$  and  $\bar{v}(\cdot) = v(\cdot)$ . Assumptions (3.1) and (3.2) are satisfied because of inequalities (3.8) and (3.9) in Lemma 3.3. Condition (3.3) is the same as (2.3). Therefore inequality (3.10), which is the same as (3.4), follows immediately because of Theorem 3.1.

#### Proof of Proposition 3.5

We have to prove (3.11) which can be written as

$$\begin{aligned} &\int_{x \in \mathbb{R}^n} \left[ \int_{2|x| < |y|} |\Omega(x-y)| h(y) dy \right]^p u(x) dx \\ &\leq (cA \|\Omega(\cdot)\|_{L^r(S^{n-1})})^p \int_{y \in \mathbb{R}^n} h^p(y) |y|^{np} v(y) dy \quad \text{for all } h(\cdot) \geq 0. \end{aligned}$$

And by duality argument, this inequality is equivalent to

$$\begin{aligned} &\int_{y \in \mathbb{R}^n} \left[ \int_{|x| < 2^{-1}|y|} |\Omega(x-y)| g(x) dx \right]^{p'} |y|^{-np'} v^{1-p'}(y) dy \\ &\leq (cA \|\Omega(\cdot)\|_{L^r(S^{n-1})})^{p'} \int_{z \in \mathbb{R}^n} g^{p'}(x) u^{1-p'}(x) dx. \quad \text{for all } g(\cdot) \geq 0 \end{aligned}$$

Therefore the task remains to get inequality (3.4) with  $t = p'$ ,  $K(x, y) = |\Omega(y - x)|$ ,  $\mathcal{K}^r = c\|\Omega(\cdot)\|_{L^r(S^{n-1})}^r$ ,  $\bar{u}(\cdot) = v^{1-p'}(\cdot)$  and  $\bar{v}(\cdot) = u^{1-p'}(\cdot)$ .

Assumptions (3.1) and (3.2) are satisfied because of inequalities (3.8) and (3.9) in Lemma 3.3. Condition (3.3) is the same as (2.3\*). Therefore inequality (3.11) follows now from Theorem 3.1.

**Proof of Lemma 3.6**

Just inequality (3.12) is proved here since the arguments needed for (3.13) are essentially the same. The case  $m = 1$ , i.e.  $1 \leq \frac{r}{t}$ , is trivial since  $\tau(t, r, n) = 0$  and consequently

$$\begin{aligned} \int_{2^{-(j+1)}|x| < |z| < 2^{-j}|x|} K^{(1-m)t'}(x, z) dz &\approx (2^{-j}|x|)^n \\ &= (2^{-j[n-\tau(t, r, n)]}|x|^n) \mathcal{K}^{(1-m)t'}. \end{aligned}$$

Therefore it can be assumed that  $m = \frac{r}{t} < 1$  for which  $(1-m)t' = \frac{t-r}{t-1} \leq 1$ . Then inequality (3.12) follows since

$$\begin{aligned} &\int_{2^{-(j+1)}|x| < |z| < 2^{-j}|x|} K^{(1-m)t'}(x, z) dz \\ &= \int_{2^{-(j+1)}|x| < |z| < 2^{-j}|x|} K^{\frac{t-r}{t-1}}(x, z) dz \\ &\leq c(2^{-j}|x|)^{n[1-\frac{t-r}{(t-1)r}]} \left( \int_{2^{-(j+1)}|x| < |z| < 2^{-j}|x|} K^r(x, z) dz \right)^{\frac{t-r}{(t-1)r}} \\ &\leq c(2^{-j}|x|)^{n[1-\frac{t-r}{(t-1)r}]} \left[ (2^{-j}|x|^n) \mathcal{K}^r \right]^{\frac{t-r}{(t-1)r}} \quad \text{by assumption (3.1)} \\ &= c(2^{-j[n-(n-1)\frac{t-r}{(t-1)r}]}|x|^n) \mathcal{K}^{\frac{t-r}{t-1}} \\ &= c(2^{-j[n-\tau(t, r, n)]}|x|^n) \mathcal{K}^{(1-m)t'}. \end{aligned}$$

**Proof of Theorem 3.1**

To get inequality (3.4) we would like to benefit from Lemma 3.2 by taking  $\varphi(z) = |z|^\varepsilon$ . And the task is reduced to check condition (3.5) (with some constant  $c(\mathcal{K}A)^t$ ).

One of the point is the following estimate of  $\Theta(x)$  which is valid for  $2^k|y| < |x| \leq 2^{k+1}|y|$ ,  $k \geq 1$ :

$$\begin{aligned} \Theta(x) &\leq c\mathcal{K}^{(1-m)t'}(2^k|y|)^{(t'-1)[- \varepsilon + n(t-1)]} \times \\ &\quad \left[ \sum_{j=1}^{\infty} 2^{-j(t'-1)[- \varepsilon + \mu(t, r, n)]} \left( \sup_{2^{-(j+1)}2^k|y| < |z| < 2^{-j}2^{k+1}|y|} \overline{v}^{(1-t')}(z) \right) \right]. \end{aligned}$$

The justification of this claim is deferred below and for the moment we give the sequel of estimates which lead to inequality (3.5). The conclusion will be obtained by using condition (3.3) since

$$\int_{2|y| < |x|} K^{mt}(x, y) \Theta^{t-1}(x) |x|^{-nt} \overline{u}(x) dx$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} \int_{2^k|y| < |x| \leq 2^{k+1}|y|} K^{mt}(x, y) \Theta^{t-1}(x) |x|^{-nt} \bar{u}(x) dx \\
&\leq c_1 \mathcal{K}^{(1-m)t} \sum_{k=1}^{\infty} \left( \int_{2^k|y| < |x| \leq 2^{k+1}|y|} K^{mt}(x, y) dx \right) \times \\
&\quad \times (2^k|y|)^{-nt} \left( \sup_{2^k|y| < |z| < 2^{k+1}|y|} \bar{u}(z) \right) (2^k|y|)^{[-\varepsilon + n(t-1)]} \times \\
&\quad \times \left[ \sum_{j=1}^{\infty} 2^{-j(t'-1)[- \varepsilon + \mu(t, r, n)]} \left( \sup_{2^{-(j+1)}2^k|y| < |z| < 2^{-j}2^{k+1}|y|} \bar{v}^{1-t'}(z) \right) \right]^{t-1} \\
&\leq c_2 \mathcal{K}^{(1-m)t} \sum_{k=1}^{\infty} \left( (2^k|y|)^n \mathcal{K}^{mt} \right) \times \quad \text{by (3.13) in Lemma 3.6} \\
&\quad \times (2^k|y|)^{-nt} \left( \sup_{2^k|y| < |z| < 2^{k+1}|y|} \bar{u}(z) \right) (2^k|y|)^{[-\varepsilon + n(t-1)]} \times \\
&\quad \times \left[ \sum_{j=1}^{\infty} 2^{-j(t'-1)[- \varepsilon + \mu(t, r, n)]} \left( \sup_{2^{-(j+1)}2^k|y| < |z| < 2^{-j}2^{k+1}|y|} \bar{v}^{(1-t')}(z) \right) \right]^{t-1} \\
&= c_2 \mathcal{K}^t |y|^{-\varepsilon} \sum_{k=1}^{\infty} 2^{-k\varepsilon} \left( \sup_{2^k|y| < |z| < 2^{k+1}|y|} \bar{u}(z) \right) \times \\
&\quad \times \left[ \sum_{j=1}^{\infty} 2^{-j(t'-1)[- \varepsilon + \mu(t, r, n)]} \left( \sup_{2^{-(j+1)}2^k|y| < |z| < 2^{-j}2^{k+1}|y|} \bar{v}^{(1-t')}(z) \right) \right]^{t-1} \\
&\leq c_2 (\mathcal{K}A)^t |y|^{-\varepsilon} \sum_{k=1}^{\infty} 2^{-k\varepsilon} \quad \text{by condition (3.3)} \\
&= c_3 (\mathcal{K}A)^t \varphi^{-1}(y).
\end{aligned}$$

It remains now to prove the above estimate claimed for  $\Theta(x)$  when  $2^k|y| < |x| < 2^{k+1}|y|$  and  $k \geq 1$ . Here (3.12) in Lemma 3.6 is useful since

$$\begin{aligned}
\Theta(x) &= \sum_{j=1}^{\infty} \int_{2^{-(j+1)}|x| < |z| < 2^{-j}|x|} K^{(1-m)t'}(x, z) \varphi^{1-t'}(z) \bar{v}^{(1-t')}(z) dz \\
&\leq c_4 \sum_{j=1}^{\infty} \left( \int_{2^{-(j+1)}|x| < |z| < 2^{-j}|x|} K^{(1-m)t'}(x, z) dz \right) \times \\
&\quad \times (2^{-j}|x|)^{(1-t')\varepsilon} \left( \sup_{2^{-(j+1)}|x| < |z| < 2^{-j}|x|} \bar{v}^{(1-t')}(z) \right) \\
&\leq c_5 \sum_{j=1}^{\infty} \left( (2^{-j[n-\tau(t, r, n)]}|x|^n) \mathcal{K}^{(1-m)t'} \right) \times
\end{aligned}$$

$$\begin{aligned}
& \times (2^{-j}|x|)^{(1-t')\varepsilon} \left( \sup_{2^{-(j+1)}|x| < |z| < 2^{-j}|x|} \overline{v}^{(1-t')}(z) \right) \\
& = c_5 \mathcal{K}^{(1-m)t'} |x|^{n+(1-t')\varepsilon} \left[ \sum_{j=1}^{\infty} 2^{-j[(1-t')\varepsilon+n-\tau(t,r,n)]} \times \right. \\
& \quad \left. \times \left( \sup_{2^{-(j+1)}|x| < |z| < 2^{-j}|x|} \overline{v}^{(1-t')}(z) \right) \right] \\
& \leq c_6 \mathcal{K}^{(1-m)t'} (2^k|y|)^{n+(1-t')\varepsilon} \left[ \sum_{j=1}^{\infty} 2^{-j[(1-t')\varepsilon+n-\tau(t,r,n)]} \times \right. \\
& \quad \left. \times \left( \sup_{2^{-(j+1)}2^k|y| < |z| < 2^{-j}2^{k+1}|y|} \overline{v}^{(1-t')}(z) \right) \right] \\
& = c_6 \mathcal{K}^{(1-m)t'} (2^k|y|)^{(t'-1)[- \varepsilon+n(t-1)]} \left[ \sum_{j=1}^{\infty} 2^{-j(t'-1)[- \varepsilon+\mu(t,r,n)]} \times \right. \\
& \quad \left. \times \left( \sup_{2^{-(j+1)}2^k|y| < |z| < 2^{-j}2^{k+1}|y|} \overline{v}^{(1-t')}(z) \right) \right].
\end{aligned}$$

### Proof of Lemma 3.7

Observe that

$$\mu(t, r, n) - n(t-1) + n\rho(t-1) = n(t-1)\rho > 0 \quad \text{for } t \leq r$$

and, by (3.15),  $r \leq t$ :

$$\mu(t, r, n) - n(t-1) + n\rho(t-1) = n(t-1) \left( \rho - \left[ 1 - \frac{1}{n} \left( 1 + \frac{(n-1)t'}{r'} \right) \right] \right) > 0.$$

Therefore one can choose a real  $\varepsilon > 0$  such that

$$-\varepsilon + \mu(t, r, n) - n(t-1) + n\rho(t-1) > 0 \quad \text{for } r > 1.$$

Combining this last information with (3.14) then condition (3.3) arises as follows

$$\begin{aligned}
& \left( \sup_{R < |x| < 2R} \overline{u}(x) \right)^{\frac{1}{t}} \left[ \sum_{j=1}^{\infty} 2^{-j(t'-1)[- \varepsilon+\mu(t,r,n)]} \left( \sup_{2^{-1}(2^{-j}R) < |y| < 2(2^{-j}R)} \overline{v}^{(1-t')}(y) \right) \right]^{\frac{1}{t'}} \\
& \leq c_1 \left( R^{-n} \int_{2^{-N}R < |x| < 2^NR} \overline{u}(x) dx \right)^{\frac{1}{t}} \left[ \sum_{j=1}^{\infty} 2^{-j(t'-1)[- \varepsilon+\mu(t,r,n)]} \times \right. \\
& \quad \left. \times \left( (2^{-j-N}R)^{-n} \int_{(2^{-j-N}R) < |y| < (2^{-j+N}R)} \overline{v}^{(1-t')}(y) dy \right) \right]^{\frac{1}{t'}}
\end{aligned}$$



$$\begin{aligned}
& \text{since } \overline{u}(\cdot), \overline{v}^{(1-t')}(\cdot) \in \mathcal{H} \\
& \leq c_2 \left( (2^N R)^{-n} \int_{|x| < 2^N R} \overline{u}(x) dx \right)^{\frac{1}{t}} \left( (2^N R)^{-n} \int_{|y| < 2^N R} \overline{v}^{(1-t')}(y) dy \right)^{\frac{1}{t'}} \times \\
& \quad \times \left[ \sum_{j=1}^{\infty} 2^{-j(t'-1)[- \varepsilon + \mu(t, r, n) - n(t-1) + n\rho(t-1)]} \right] \\
& \text{because } \overline{v}^{(1-t')}(\cdot) \in RD_{\rho} \\
& \leq c_3 A \quad \text{by condition (3.14).}
\end{aligned}$$

### Acknowledgement

The author would like to thank the referee for having pointed some corrections to bring about the preliminary version of this paper.

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